MATH 8610 (SPRING 2019) HOMEWORK 1

Assigned 01/15/18, due 01/29/18 (Tuesday) 5pm.

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1. **[Q1] (10 pts)** (a) Find the absolute and relative condition numbers of $f(x) = e^{-2x}$ and $f(x) = \ln^3 x$. For what values of x are these functions sensitive to perturbations? (b) Let $x_1,\ x_2 \in \mathbb{R}^+$, and $f(x_1, x_2) = x_1^{x_2}$. Find the relative condition number of $f(x)$, and for what range of values of x_1 and x_2 is the problem ill-conditioned.

2. **[Q2] (10 pts)** Consider the recurrence $x_{k+1} = 111 - \frac{1130 - \frac{3000}{x_{k-1}}}{x_k}$, whose general solution is $x_k = \frac{100^{k+1}a+6^{k+1}b+5^{k+1}c}{100^k a+6^k b+5^k c}$, where a, b and c depend on the initial values. Given $x_0 = \frac{11}{2}$ and $x_1 = \frac{61}{11}$, we have $a = 0$, $b = c = 1$.
(a) Show that this gives a monotonically increasing sequence to 6.
(b) Implement this recurrence on MATLAB, plot $\{x_k\}$, compare with the exact solution. What is the condition number of the limit of this particular sequence as a function of x_0 and x_1?

3. **[Q3] (10 pts)** Let $p_{24}(x) = (x-1)(x-2)\cdots(x-24) = a_0 + a_1x + \cdots + a_{23}x^{23} + a_{24}x^{24}$, where $a_{16} \approx 2.9089 \times 10^{14}$, $a_{17} \approx -1.2191 \times 10^{13}$, $a_{18} \approx 4.1491 \times 10^{11}$, $a_{19} \approx -1.1277 \times 10^{10}$, and $a_{20} \approx 2.3881 \times 10^8$. Evaluate the relative condition number of the k-th root $x_k = k$ subject to the perturbation of a_k for $k = 16$ to 20 and find the root that is most sensitive to the perturbation of the corresponding coefficient. Use the attached MATLAB data file `wilk24mc.mat` containing the coefficients $a_{24}, a_{23}, \ldots, a_1$, and use MATLAB's `roots` to find the roots. Compare with the true roots and comment on what you see.

4. **[Q4] (10 pts)** Let $x_0, x_1, \ldots, x_n$ be $n+1$ equidistant points on $[-1, 1]$, where $x_0 = -1$ and $x_n = 1$. Use MATLAB's `vander` to generate Vandermonde matrices A for $n = 9, 19, 29, 39$. Let $x = [1\ 1\ \ldots\ 1]^T$ and $b = Ax$. Pretend that we do not know x and use numerical algorithms to solve for x. Let $\hat{x}$ be the computed solution.
Compute the relative forward errors $\frac{\|\hat{x}-x\|}{\|x\|}$ and the smallest relative backward errors $\frac{\|b-A\hat{x}\|_2}{\|A\|_2\|\hat{x}\|_2} = \min\left\{\frac{\|\Delta A\|_2}{\|A\|_2} : (A+\Delta A)\hat{x} = b\right\}$ for (a) GEPP (MATLAB's backslash), (b) QR factorization of A, (c) Cramer's rule, (d) A^{-1} multiplied with b, and (e) GE without pivoting. Comment on the forward/backward stability of these methods.

5. **[Q5] (20 pts)** Though pivoting is needed for factorizing general matrices, it is not needed for symmetric positive definite and diagonally dominant matrices.
(a) For a symmetric positive definite A, with the one-step Cholesky factorization
$$A = \begin{bmatrix} a_{11} & w^T \\ w & K \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \frac{w}{\sqrt{a_{11}}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K - \frac{ww^T}{a_{11}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \frac{w^T}{\sqrt{a_{11}}} \\ 0 & I \end{bmatrix} = R_1^T A_1 R_1,$$
show that the submatrix $K - \frac{ww^T}{a_{11}}$ is symmetric positive definite. Consequently, the factorization can be completed without break-down. Then, show that $\|R\|_2 = \|A\|_2^{\frac{1}{2}}$, which means the element in R are uniformly bounded by that of $\|A\|$. Explain why this observation leads to the backward stability of Cholesky factorization.

(b) Suppose that $A = \begin{bmatrix} \alpha & w^T \\ v & C \end{bmatrix}$ is column diagonally dominant, with one-step LU factorization $A = \begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C - \frac{1}{\alpha} v w^T \end{bmatrix} \begin{bmatrix} \alpha & w^T \\ 0 & I \end{bmatrix}$. Show that the submatirx $C - \frac{1}{\alpha} v w^T$ is also column diagonally dominant, and no pivoting is needed.

(c) Show that the worst-case growth factor $\rho_n = 2^{n-1}$ for GEPP. Compared to $\rho_n \leq C n^{\frac{1}{2}+\frac{1}{4}\ln n}$ with complete pivoting and $\rho_n \leq 1.5 n^{\frac{3}{4}\ln n}$ with *rook pivoting*, this is much larger. However, we construct matrices with random elements, each are independent samples from the normal distribution of mean 0 and standard deviation $\frac{1}{\sqrt{n}}$ (`A = randn(n,n)/sqrt(n)`). Let $n = 32, 64, \ldots, 2048$, and for each n, repeat the experiment 5000 times. Note that $\rho_n = \frac{\max_{ij} |u_{ij}|}{\max_{ij} |a_{ij}|}$ for LUPP. Find the percent of experiments when $\rho_n > \sqrt{n}$. Comment on the chance of having a large ρ_n.